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On a third-order multi-point boundary value problem at resonance ☆

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Abstract

In this paper, we prove some existence results for a third order multi-point boundary value problem at resonance. Our method is based upon the coincidence degree theory of Mawhin.

Keywords: Multi-point boundary value problem; Resonance; Coincidence degree theory; Third-order differential equations

1. Introduction

This paper deals with the following third-order ordinary differential equation:

$$x'''(t) = f\big(t, x(t), x'(t), x''(t)\big) + e(t), \quad t \in (0,1), \tag{1.1}$$

with the following boundary value conditions:

$$x(0) = \sum_{i=1}^{m-2} \alpha_i x(\xi_i), \qquad x'(0) = 0, \qquad x(1) = \beta x(\eta). \tag{1.2}$$

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Here $f:[0,1]\times R^3 \to R$ is a continuous function, $e \in L^1[0,1]$, α_i $(1 \leqslant i \leqslant m-2) \in R$, $\beta \geqslant 0$, $0 < \xi_1 < \xi_2 < \cdots < \xi_{m-2} < 1$, and $\eta \in (0,1)$.

Similarly in [1,2], for certain boundary condition case such that the linear operator $Lx = x'''(t) = 0$, defined in a suitable Banach space, is invertible, the so-called non-resonance case. Otherwise, the so-called resonance case.

For the non-resonance case, we refer to see [3,10] and the references therein.

For the resonance case, the boundary value problem is approached in several ways. Such as, Ma [7] studied existence and multiplicity results for the boundary value problem

$$x''' + k^2 x' + g(x, x') = p(t), \qquad x'(0) = x'(\pi) = x(\eta) = 0. \tag{1.3}$$

by combining the Lyapunov–Schmit procedure with the continuum theory for O-epi maps. In the case $k = 1$, the solvability of (1.3) has been considered by Nagle and Pothoven [9] under the condition that g is bounded on one side. But the more classical method is to decompose the space in the form of a direct sum of subspaces, one of which is $\operatorname{Ker} L$, and then to work with the corresponding projections on these spaces. For instance, Feng [1], Gupta [4,5], and Liu and Yu [6] used this method to study the existence results for some second order multi-point boundary value problems at resonance case.

Inspired by the work of the above papers, in the present article, we use the coincidence degree theory of Mawhin [8] to discuss the existence of solution for third-order multi-point BVP (1.1), (1.2) at resonance case, and establish some existence theorems under nonlinear growth restriction of f.

2. Existence results

First we present some preliminaries needed to understand how the fixed point result of Mawhin [8] is concerned.

Let Y, Z be real Banach spaces and let $L : \operatorname{dom} L \subset Y \to Z$ be a linear operator which is Fredholm map of index zero and $P : Y \to Y$, $Q : Z \to Z$ be continuous projectors such that $\operatorname{Im} P = \operatorname{Ker} L$, $\operatorname{Ker} Q = \operatorname{Im} L$ and $Y = \operatorname{Ker} L \oplus \operatorname{Ker} P$, $Z = \operatorname{Im} L \oplus \operatorname{Im} Q$. It follows that $L|_{\operatorname{dom} L \cap \operatorname{Ker} P} : \operatorname{dom} L \cap \operatorname{Ker} P \to \operatorname{Im} L$ is invertible, we denote the inverse of that map by K_P. Let Ω be an open bounded subset of Y such that $\operatorname{dom} L \cap \Omega \neq \emptyset$, the map $N : Y \to Z$ is said to be L-compact on $\bar{\Omega}$ if the map $QN(\bar{\Omega})$ is bounded and $K_P(I-Q)N : \bar{\Omega} \to Y$ is compact. For more details we refer the reader to the lecture notes of Mawhin [8].

To obtain our existence results we use the following fixed point theorem of Mawhin [8].

Theorem 2.1. *Let L be a Fredholm operator of index zero and let N be L-compact on $\bar{\Omega}$. Assume that the following conditions are satisfied*:

(i) $Lx \neq \lambda Nx$ *for every* $(x, \lambda) \in [(\operatorname{dom} L \setminus \operatorname{Ker} L) \cap \partial\Omega] \times (0,1)$.
(ii) $Nx \notin \operatorname{Im} L$ *for every* $x \in \operatorname{Ker} L \cap \partial\Omega$.
(iii) $\deg(QN|_{\operatorname{Ker} L}, \Omega \cap \ker L, 0) \neq 0$, *where* $Q : Z \to Z$ *is a projection as above with* $\operatorname{Im} L = \operatorname{Ker} Q$.

Then the equation $Lx = Nx$ *has at least one solution in* $\operatorname{dom} L \cap \bar{\Omega}$.

In the following, we shall use the classical spaces $C[0,1]$, $C^1[0,1]$, $C^2[0,1]$, and $L^1[0,1]$. For $x \in C^2[0,1]$, we use the norm $\|x\|_\infty = \max_{t\in[0,1]}|x(t)|$ and $\|x\| = \max\{\|x\|_\infty, \|x'\|_\infty, \|x''\|_\infty\}$, and denote the norm in $L^1[0,1]$ by $\|\cdot\|_1$. We will use the Sobolev space $W^{3,1}(0,1)$ which may be defined by

$$W^{3,1}(0,1) = \big\{x:[0,1]\to R\colon\ x, x', x'' \text{ are absolutely continuous on } [0,1] \\ \text{with } x''' \in L^1[0,1]\big\}.$$

Now we prove existence results for BVP (1.1), (1.2) in the following cases:

(i) $\beta = 1$, $\sum_{i=1}^{m-2}\alpha_i = 1$, $\sum_{i=1}^{m-2}\alpha_i\xi_i^2 = 0$;
(ii) $\beta = 1/\eta^2$, $\sum_{i=1}^{m-2}\alpha_i = 1$, $\sum_{i=1}^{m-2}\alpha_i\xi_i^2 = 0$.

Let $Y = C^2[0,1]$, $Z = L^1[0,1]$, L is the linear operator from $\operatorname{dom} L \subset Y$ to Z with

$$\operatorname{dom} L = \left\{x \in W^{3,1}(0,1)\colon\ x(0) = \sum_{i=1}^{m-2}\alpha_i x(\xi_i),\ x'(0) = 0,\ x(1) = \beta x(\eta)\right\}$$

and $Lx = x'''$, $x \in \operatorname{dom} L$. We define $N : Y \to Z$ by setting

$$Nx = f\big(t, x(t), x'(t), x''(t)\big) + e(t), \quad t \in (0,1).$$

Then BVP (1.1), (1.2) can be written as $Lx = Nx$.

Lemma 2.1. *If $\sum_{i=1}^{m-2}\alpha_i = 1$, then there exists $l \in \{0,1,\dots,m-3\}$, such that*

$$\sum_{i=1}^{m-2}\alpha_i\xi_i^{l+3} \neq 0.$$

Proof. Suppose the assertion fails to be true, then

$$\sum_{i=1}^{m-2}\alpha_i\xi_i^{l+3} = 0, \quad l = 0,1,\dots,m-3,$$

we have

$$\begin{pmatrix} \xi_1^3 & \xi_2^3 & \cdots & \xi_{m-2}^3 \\ \xi_1^4 & \xi_2^4 & \cdots & \xi_{m-2}^4 \\ \vdots & \vdots & \ddots & \vdots \\ \xi_1^m & \xi_2^m & \cdots & \xi_{m-2}^m \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_{m-2} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Since

$$\det\begin{pmatrix} \xi_1^3 & \xi_2^3 & \cdots & \xi_{m-2}^3 \\ \xi_1^4 & \xi_2^4 & \cdots & \xi_{m-2}^4 \\ \vdots & \vdots & \ddots & \vdots \\ \xi_1^m & \xi_2^m & \cdots & \xi_{m-2}^m \end{pmatrix} = \xi_1^3\xi_2^3\cdots\xi_{m-2}^3 \begin{vmatrix} 1 & 1 & \cdots & 1 \\ \xi_1 & \xi_2 & \cdots & \xi_{m-2} \\ \vdots & \vdots & \ddots & \vdots \\ \xi_1^{m-3} & \xi_2^{m-3} & \cdots & \xi_{m-2}^{m-3} \end{vmatrix}$$

$$= \left(\prod_{l=1}^{m-2} \xi_l^3\right) \prod_{1 \leqslant i < j \leqslant m-2} (\xi_j - \xi_i) \neq 0,$$

then we have

$$\alpha_1 = \alpha_2 = \cdots = \alpha_{m-2} = 0,$$

which is a contradiction from $\sum_{i=1}^{m-2} \alpha_i = 1$. Therefore Lemma 2.1 holds. □

Lemma 2.2. *If $\beta = 1$, $\sum_{i=1}^{m-2} \alpha_i = 1$, $\sum_{i=1}^{m-2} \alpha_i \xi_i^2 = 0$, then $L : \operatorname{dom} L \subset Y \to Z$ is a Fredholm operator of index zero. Furthermore, the linear continuous projector operator $Q : Z \to Z$ can be defined by*

$$Qy = \frac{(l+1)(l+2)(l+3)}{\sum_{i=1}^{m-2} \alpha_i \xi_i^{l+3}} \left(\sum_{i=1}^{m-2} \alpha_i \int_0^{\xi_i} \int_0^s \int_0^\tau y(v)\, dv\, d\tau\, ds\right) t^l,$$

and the linear operator $K_P : \operatorname{Im} L \to \operatorname{dom} L \cap \operatorname{Ker} P$ can be written by

$$K_P y = -\frac{t^2}{1-\eta^2} \int_\eta^1 \int_0^s \int_0^\tau y(v)\, dv\, d\tau\, ds + \int_0^t \int_0^s \int_0^\tau y(v)\, dv\, d\tau\, ds.$$

Furthermore

$$\|K_P y\| \leqslant \Delta_1 \|y\|_1, \quad \textit{for all } y \in \operatorname{Im} L,$$

where $\Delta_1 = (2/(1+\eta)) + 1$.

Proof. It is clear that

$$\operatorname{Ker} L = \{x \in \operatorname{dom} L\colon\ x = c,\ c \in R\}.$$

Now we show that

$$\operatorname{Im} L = \left\{ y \in Z\colon\ \sum_{i=1}^{m-2} \alpha_i \int_0^{\xi_i} \int_0^s \int_0^\tau y(v)\, dv\, d\tau\, ds = 0 \right\}. \tag{2.1}$$

Since the problem

$$x''' = y \tag{2.2}$$

has a solution $x(t)$ satisfied $x(0) = \sum_{i=1}^{m-2} \alpha_i x(\xi_i)$, $x'(0) = 0$, $x(1) = \beta x(\eta)$, if and only if

$$\sum_{i=1}^{m-2} \alpha_i \int_0^{\xi_i} \int_0^s \int_0^\tau y(v)\, dv\, d\tau\, ds = 0. \tag{2.3}$$

In fact, if (2.2) has solution $x(t)$ satisfied $x(0) = \sum_{i=1}^{m-2} \alpha_i x(\xi_i)$, $x'(0) = 0$, $x(1) = \beta x(\eta)$, then from (2.2) we have

$$x(t) = x(0) + x'(0)t + \frac{1}{2}x''(0)t^2 + \int_0^t \int_0^s \int_0^\tau y(v)\,dv\,d\tau\,ds$$
$$= x(0) + \frac{1}{2}x''(0)t^2 + \int_0^t \int_0^s \int_0^\tau y(v)\,dv\,d\tau\,ds.$$

According to $\sum_{i=1}^{m-2} \alpha_i = 1$, $\sum_{i=1}^{m-2} \alpha_i \xi_i^2 = 0$, we obtain

$$\sum_{i=1}^{m-2} \alpha_i \int_0^{\xi_i} \int_0^s \int_0^\tau y(v)\,dv\,d\tau\,ds = 0.$$

On the other hand, if (2.3) holds, setting

$$x(t) = c - \frac{t^2}{1-\eta^2} \int_\eta^1 \int_0^s \int_0^\tau y(v)\,dv\,d\tau\,ds + \int_0^t \int_0^s \int_0^\tau y(v)\,dv\,d\tau\,ds,$$

where c is an arbitrary constant, then $x(t)$ is a solution of (2.2), and $x(0) = \sum_{i=1}^{m-2} \alpha_i x(\xi_i)$, $x'(0) = 0$, $x(1) = \beta x(\eta)$. Hence (2.1) holds.

For $y \in Z$, taking the projector

$$Qy = \frac{(l+1)(l+2)(l+3)}{\sum_{i=1}^{m-2} \alpha_i \xi_i^{l+3}} \left(\sum_{i=1}^{m-2} \alpha_i \int_0^{\xi_i} \int_0^s \int_0^\tau y(v)\,dv\,d\tau\,ds \right) t^l.$$

Let $y_1 = y - Qy$, we obtain that

$$\sum_{i=1}^{m-2} \alpha_i \int_0^{\xi_i} \int_0^s \int_0^\tau y_1(v)\,dv\,d\tau\,ds = 0,$$

then $y_1 \in \operatorname{Im} L$. Hence $Z = \operatorname{Im} L + R$, since $\operatorname{Im} L \cap R = \{0\}$, we have $Z = \operatorname{Im} L \oplus R$, thus

$$\dim \operatorname{Ker} L = \dim R = \operatorname{co}\dim \operatorname{Im} L = 1.$$

Hence L is a Fredholm operator of index zero.

Taking $P : Y \to Y$ as follows:

$$Px = x(0),$$

then the generalized inverse $K_P : \operatorname{Im} L \to \operatorname{dom} L \cap \operatorname{Ker} P$ of L can be written by

$$K_P y = -\frac{t^2}{1-\eta^2} \int_\eta^1 \int_0^s \int_0^\tau y(v)\,dv\,d\tau\,ds + \int_0^t \int_0^s \int_0^\tau y(v)\,dv\,d\tau\,ds.$$

In fact, for $\forall y \in \operatorname{Im} L$, we have

$$(LK_P)y(t) = \left[(K_P y)(t)\right]''' = y(t),$$

and for $x \in \operatorname{dom} L \cap \operatorname{Ker} P$, we know

$$(K_P L)x(t) = -\frac{t^2}{1-\eta^2} \int_\eta^1 \int_0^s \int_0^\tau x'''(v)\,dv\,d\tau\,ds + \int_0^t \int_0^s \int_0^\tau x'''(v)\,dv\,d\tau\,ds$$
$$= -\frac{t^2}{1-\eta^2}\left[x(1) - x(\eta) - x'(0)(1-\eta) - \frac{1}{2}x''(0)\left(1-\eta^2\right)\right]$$
$$+ x(t) - x(0) - x'(0)t - \frac{1}{2}x''(0)t^2,$$

in view of $x \in \operatorname{dom} L \cap \operatorname{Ker} P$, $x'(0) = 0$, $x(1) = x(\eta)$, and $Px = x(0) = 0$, thus

$$(K_P L)x(t) = x(t).$$

This shows that $K_P = (L|_{\operatorname{dom} L \cap \operatorname{Ker} P})^{-1}$. Also we have

$$\|K_P y\|_\infty \leqslant \frac{1}{1-\eta^2} \int_\eta^1 \int_0^s \int_0^\tau \left|y(v)\right| dv\,d\tau\,ds + \int_0^1 \int_0^1 \int_0^1 \left|y(v)\right| dv\,d\tau\,ds$$
$$\leqslant \left(\frac{1}{1+\eta} + 1\right)\|y\|_1 \leqslant \Delta_1 \|y\|_1,$$

and from

$$(K_P y)'(t) = -\frac{2t}{1-\eta^2} \int_\eta^1 \int_0^s \int_0^\tau y(v)\,dv\,d\tau\,ds + \int_0^t \int_0^\tau y(v)\,dv\,d\tau,$$

$$(K_P y)''(t) = -\frac{2}{1-\eta^2} \int_\eta^1 \int_0^s \int_0^\tau y(v)\,dv\,d\tau\,ds + \int_0^t y(v)\,dv,$$

we obtain

$$\left\|(K_P y)'\right\|_\infty \leqslant \frac{2}{1-\eta^2} \int_\eta^1 \int_0^1 \int_0^1 \left|y(v)\right| dv\,d\tau\,ds + \int_0^1 \int_0^1 \left|y(v)\right| dv\,d\tau = \Delta_1 \|y\|_1,$$

$$\left\|(K_P y)''\right\|_\infty \leqslant \frac{2}{1-\eta^2} \int_\eta^1 \int_0^1 \int_0^1 \left|y(v)\right| dv\,d\tau\,ds + \int_0^1 \left|y(v)\right| dv = \Delta_1 \|y\|_1,$$

then $\|K_P y\| \leqslant \Delta_1 \|y\|_1$. This completes the proof of Lemma 2.2. □

Theorem 2.2. *Let $f : [0,1] \times R^3 \to R$ be a continuous function, assume that*

(A_1) *There exist functions $a, b, c, d, r \in L^1[0,1]$, and constant $\theta \in [0,1)$, such that for all $(x,y,z) \in R^3$, $t \in [0,1]$, either*

$$\left|f(t,x,y,z)\right| \leqslant a(t)|x| + b(t)|y| + c(t)|z| + d(t)|z|^\theta + r(t), \tag{2.4}$$

or

$$\left|f(t,x,y,z)\right| \leqslant a(t)|x| + b(t)|y| + c(t)|z| + d(t)|y|^{\theta} + r(t), \tag{2.5}$$

or else

$$\left|f(t,x,y,z)\right| \leqslant a(t)|x| + b(t)|y| + c(t)|z| + d(t)|x|^{\theta} + r(t). \tag{2.6}$$

(A_2) *There exists a constant $M > 0$, such that for $x \in \operatorname{dom} L$, if $|x(t)| > M$, for all $t \in [0,1]$, then*

$$\sum_{i=1}^{m-2} \alpha_i \int_0^{\xi_i} \int_0^{s} \int_0^{\tau} \left[f\left(v, x(v), x'(v), x''(v)\right) + e(v) \right] dv\, d\tau\, ds \neq 0. \tag{2.7}$$

(A_3) *There exists a constant $M^* > 0$, such that for $c \in R$, if $|c| > M^*$, then either*

$$c \cdot \sum_{i=1}^{m-2} \alpha_i \int_0^{\xi_i} \int_0^{s} \int_0^{\tau} \left[f(v, c, 0, 0) + e(v) \right] dv\, d\tau\, ds < 0, \tag{2.8}$$

or else

$$c \cdot \sum_{i=1}^{m-2} \alpha_i \int_0^{\xi_i} \int_0^{s} \int_0^{\tau} \left[f(v, c, 0, 0) + e(v) \right] dv\, d\tau\, ds > 0. \tag{2.9}$$

Then for every $e \in L^1[0,1]$, the BVP (1.1), (1.2) with $\beta = 1$, $\sum_{i=1}^{m-2} \alpha_i = 1$, $\sum_{i=1}^{m-2} \alpha_i \xi_i^2 = 0$ has at least one solution in $C^2[0,1]$ provided that

$$\|a\|_1 + \|b\|_1 + \|c\|_1 < \frac{1}{\Delta_2},$$

where $\Delta_2 = \Delta_1 + 1$, Δ_1 as in Lemma 2.2.

Proof. We divide the proof into the following steps.

Step 1. *Let*

$$\Omega_1 = \left\{ x \in \operatorname{dom} L \setminus \operatorname{Ker} L\colon\ Lx = \lambda N x \text{ for some } \lambda \in [0,1] \right\}.$$

Then Ω_1 is bounded.

Suppose that $x \in \Omega_1$, $Lx = \lambda Nx$, thus $\lambda \neq 0$, $Nx \in \operatorname{Im} L = \operatorname{Ker} Q$, hence

$$\sum_{i=1}^{m-2} \alpha_i \int_0^{\xi_i} \int_0^{s} \int_0^{\tau} \left[f\left(v, x(v), x'(v), x''(v)\right) + e(v) \right] dv\, d\tau\, ds = 0,$$

thus, from (A_2), there exists $t_0 \in [0,1]$, such that $|x(t_0)| < M$, in view of $x(0) = x(t_0) - \int_0^{t_0} x'(t)\, dt$, we obtain

$$\left|x(0)\right| \leqslant M + \|x'\|_\infty. \tag{2.10}$$

From $x(1)=\beta x(\eta)=x(\eta)$, there exists $t_1\in(\eta,1)$, such that $x'(t_1)=0$, thus from $x'(t)=x'(t_1)+\int_{t_1}^{t}x''(t)\,dt$, one has

$$\|x'\|_\infty\leqslant\|x''\|_1. \tag{2.11}$$

Again from $x'(0)=x'(t_1)=0$, there exists $t_2\in(0,t_1)$, such that $x''(t_2)=0$, thus from $x''(t)=x''(t_2)+\int_{t_2}^{t}x'''(t)\,dt$, we obtain

$$\|x''\|_\infty\leqslant\|x'''\|_1, \tag{2.12}$$

hence from (2.10), (2.11), and (2.12), we have

$$\begin{aligned}\|Px\|=\bigl|x(0)\bigr|&\leqslant M+\|x'\|_\infty\leqslant M+\|x''\|_1\leqslant M+\|x''\|_\infty\\&\leqslant M+\|x'''\|_1=M+\|Lx\|_1\leqslant M+\|Nx\|_1.\end{aligned} \tag{2.13}$$

Again for $x\in\Omega_1$, $x\in\operatorname{dom}L\setminus\operatorname{Ker}L$, then $(I-P)x\in\operatorname{dom}L\cap\operatorname{Ker}P$, $LPx=0$, thus from Lemma 2.2, we know

$$\begin{aligned}\bigl\|(I-P)x\bigr\|=\bigl\|K_PL(I-P)x\bigr\|&\leqslant\Delta_1\bigl\|L(I-P)x\bigr\|_1=\Delta_1\|Lx\|_1\\&\leqslant\Delta_1\|Nx\|_1.\end{aligned} \tag{2.14}$$

From (2.13) and (2.14), we have

$$\|x\|\leqslant\|Px\|+\bigl\|(I-P)x\bigr\|=\bigl|x(0)\bigr|+\bigl\|(I-P)x\bigr\|\leqslant\Delta_2\|Nx\|_1+M. \tag{2.15}$$

If (2.4) holds, then from (2.15), we obtain

$$\begin{aligned}\|x\|\leqslant\Delta_2\biggl[&\|a\|_1\|x\|_\infty+\|b\|_1\|x'\|_\infty+\|c\|_1\|x''\|_\infty+\|d\|_1\|x''\|_\infty^\theta+\|r\|_1+\|e\|_1\\&+\frac{M}{\Delta_2}\biggr].\end{aligned} \tag{2.16}$$

Thus, from $\|x\|_\infty\leqslant\|x\|$ and (2.16), we have

$$\begin{aligned}\|x\|_\infty\leqslant\frac{\Delta_2}{1-\Delta_2\|a\|_1}\biggl[&\|b\|_1\|x'\|_\infty+\|c\|_1\|x''\|_\infty+\|d\|_1\|x''\|_\infty^\theta+\|r\|_1+\|e\|_1\\&+\frac{M}{\Delta_2}\biggr].\end{aligned} \tag{2.17}$$

From $\|x'\|_\infty\leqslant\|x\|$, (2.16) and (2.17), one has

$$\begin{aligned}\|x'\|_\infty&\leqslant\|x\|\\&\leqslant\Delta_2\biggl[1+\frac{\Delta_2\|a\|_1}{1-\Delta_2\|a\|_1}\biggr]\\&\quad\times\biggl[\|b\|_1\|x'\|_\infty+\|c\|_1\|x''\|_\infty+\|d\|_1\|x''\|_\infty^\theta+\|r\|_1+\|e\|_1+\frac{M}{\Delta_2}\biggr]\\&=\frac{\Delta_2}{1-\Delta_2\|a\|_1}\\&\quad\times\biggl[\|b\|_1\|x'\|_\infty+\|c\|_1\|x''\|_\infty+\|d\|_1\|x''\|_\infty^\theta+\|r\|_1+\|e\|_1+\frac{M}{\Delta_2}\biggr],\end{aligned}$$

$$\|x'\|_\infty \leqslant \frac{\Delta_2}{1-\Delta_2(\|a\|_1+\|b\|_1)} \times \left[\|c\|_1\|x''\|_\infty + \|d\|_1\|x''\|_\infty^\theta + \|r\|_1 + \|e\|_1 + \frac{M}{\Delta_2}\right]. \tag{2.18}$$

Again from $\|x''\|_\infty \leqslant \|x\|$, (2.16), (2.17), and (2.18), we get

$$\begin{aligned}
\|x''\|_\infty \leqslant \|x\| &\leqslant \left[\Delta_2\|b\|_1 + \frac{\Delta_2^2\|a\|_1\|b\|_1}{1-\Delta_2\|a\|_1}\right]\|x'\|_\infty \\
&\quad + \left[\frac{\Delta_2^2\|a\|_1}{1-\Delta_2\|a\|_1} + \Delta_2\right] \\
&\quad \times \left[\|c\|_1\|x''\|_\infty + \|d\|_1\|x''\|_\infty^\theta + \|r\|_1 + \|e\|_1 + \frac{M}{\Delta_2}\right] \\
&\leqslant \left[\frac{\Delta_2^2\|b\|_1}{(1-\Delta_2(\|a\|_1+\|b\|_1))(1-\Delta_2\|a\|_1)} + \frac{\Delta_2}{1-\Delta_2\|a\|_1}\right] \\
&\quad \times \left[\|c\|_1\|x''\|_\infty + \|d\|_1\|x''\|_\infty^\theta + \|r\|_1 + \|e\|_1 + \frac{M}{\Delta_2}\right] \\
&= \frac{\Delta_2}{1-\Delta_2(\|a\|_1+\|b\|_1)} \\
&\quad \times \left[\|c\|_1\|x''\|_\infty + \|d\|_1\|x''\|_\infty^\theta + \|r\|_1 + \|e\|_1 + \frac{M}{\Delta_2}\right],
\end{aligned}$$

$$\|x''\|_\infty \leqslant \frac{\Delta_2\|d\|_1}{1-\Delta_2(\|a\|_1+\|b\|_1+\|c\|_1)}\|x''\|_\infty^\theta + \frac{\Delta_2}{1-\Delta_2(\|a\|_1+\|b\|_1+\|c\|_1)}\left[\|r\|_1 + \|e\|_1 + \frac{M}{\Delta_2}\right], \tag{2.19}$$

since $\theta \in [0, 1)$, from (2.19), there exist $M_1 > 0$, such that

$$\|x''\|_\infty \leqslant M_1, \tag{2.20}$$

thus from (2.20) and (2.18), there exist $M_2 > 0$, such that

$$\|x'\|_\infty \leqslant M_2, \tag{2.21}$$

therefore from (2.21) and (2.17), there exist $M_3 > 0$, such that

$$\|x\|_\infty \leqslant M_3, \tag{2.22}$$

hence

$$\|x\| = \max\bigl\{\|x\|_\infty, \|x'\|_\infty, \|x''\|_\infty\bigr\} \leqslant \max\{M_1, M_2, M_3\}.$$

Again from (2.4), (2.20), (2.21), and (2.22), we have

$$\begin{aligned}
\|x'''\|_1 &= \|Lx\|_1 \leqslant \|Nx\|_1 \\
&\leqslant \|a\|_1 M_3 + \|b\|_1\|M_2 + \bigl(\|c\|_1 + \|d\|_1\bigr)M_1 + \|r\|_1 + \|e\|_1 := M_4.
\end{aligned}$$

Hence we show that Ω_1 is bounded.

If (2.5) or (2.6) holds, similar to the above argument, we can prove Ω_1 is bounded too.

Step 2. *The set $\Omega_2 = \{x \in \operatorname{Ker} L\colon\ Nx \in \operatorname{Im} L\}$ is bounded.*

Let $x \in \Omega_2$, $x \in \operatorname{Ker} L = \{x \in \operatorname{dom} L\colon\ x = c,\ c \in R\}$, and $QNx = 0$, thus,

$$\sum_{i=1}^{m-2} \alpha_i \int_0^{\xi_i} \int_0^{s} \int_0^{\tau} \big[f(v, c, 0, 0) + e(v)\big]\, dv\, d\tau\, ds = 0.$$

From (A_2), $\|x\|_\infty = |c| \leqslant M$, so $\|x\| = |c| \leqslant M$, thus Ω_2 is bounded.

Step 3. *If the first part of the condition (A_3) holds, that is, there exists $M^* > 0$, such that for any $c \in R$, if $|c| > M^*$, then*

$$c \cdot \frac{(l+1)(l+2)(l+3)}{\sum_{i=1}^{m-2} \alpha_i \xi_i^{l+3}} \left(\sum_{i=1}^{m-2} \alpha_i \int_0^{\xi_i} \int_0^{s} \int_0^{\tau} \big[f(v, c, 0, 0) + e(v)\big]\, dv\, d\tau\, ds \right) t^l < 0. \tag{2.23}$$

Let

$$\Omega_3 = \big\{x \in \operatorname{Ker} L\colon\ -\lambda J x + (1-\lambda) QNx = 0,\ \lambda \in [0,1]\big\},$$

here $J : \operatorname{Ker} L \to \operatorname{Im} Q$ is the linear isomorphism given by $J(c) = c$, $\forall c \in R$. Then Ω_3 is bounded.

Since for $x = c_0 \in \Omega_3$, then we obtain

$$\lambda c_0 = (1-\lambda) \cdot \frac{(l+1)(l+2)(l+3)}{\sum_{i=1}^{m-2} \alpha_i \xi_i^{l+3}} \times \left(\sum_{i=1}^{m-2} \alpha_i \int_0^{\xi_i} \int_0^{s} \int_0^{\tau} \big[f(v, c, 0, 0) + e(v)\big]\, dv\, d\tau\, ds \right) t^l.$$

If $\lambda = 1$, then $c_0 = 0$. Otherwise, if $|c_0| > M^*$, in view of (2.23), one has

$$\lambda c_0^2 = c_0 \cdot (1-\lambda) \cdot \frac{(l+1)(l+2)(l+3)}{\sum_{i=1}^{m-2} \alpha_i \xi_i^{l+3}} \times \left(\sum_{i=1}^{m-2} \alpha_i \int_0^{\xi_i} \int_0^{s} \int_0^{\tau} \big[f(v, c, 0, 0) + e(v)\big]\, dv\, d\tau\, ds \right) t^l < 0,$$

which contradicts $\lambda c_0^2 \geqslant 0$. Then $|x| = |c_0| \leqslant M^*$, we obtain $\|x\| \leqslant M^*$, thus $\Omega_3 \subset \{x \in \operatorname{Ker} L\colon\ \|x\| \leqslant M^*\}$ is bounded.

Step 4. *If the second part of the condition (A_3) holds, that is, there exists $M^* > 0$, such that for any $c \in R$, if $|c| > M^*$, then*

$$c \cdot \frac{(l+1)(l+2)(l+3)}{\sum_{i=1}^{m-2} \alpha_i \xi_i^{l+3}} \left(\sum_{i=1}^{m-2} \alpha_i \int_0^{\xi_i} \int_0^{s} \int_0^{\tau} \big[f(v, c, 0, 0) + e(v)\big]\, dv\, d\tau\, ds \right) t^l > 0. \tag{2.24}$$

Let

$$\Omega_3 = \bigl\{x \in \operatorname{Ker} L\colon\ \lambda Jx + (1-\lambda)QNx = 0,\ \lambda \in [0,1]\bigr\},$$

here J as in Step 3*. Similar to the above argument, we can verify that Ω_3 is bounded.*

In the following, we shall prove that all the conditions of Theorem 2.1 are satisfied.

Let Ω be a bounded open subset of Y such that $\bigcup_{i=1}^{3} \bar{\Omega}_i \subset \Omega$. By the Ascoli–Arzela theorem, we can show that $K_P(I-Q)N : \bar{\Omega} \to Y$ is compact, thus N is L-compact on $\bar{\Omega}$. Then by the above argument, we have

(i) $Lx \neq \lambda Nx$ for every $(x,\lambda) \in [(\operatorname{dom} L \setminus \operatorname{Ker} L) \cap \partial\Omega] \times (0,1)$.
(ii) $Nx \notin \operatorname{Im} L$ for every $x \in \operatorname{Ker} L \cap \partial\Omega$.
(iii) Let

$$H(x,\lambda) = \pm\lambda Jx + (1-\lambda)QNx.$$

According to the above argument, we know $H(x,\lambda) \neq 0$, for $x \in \operatorname{Ker} L \cap \partial\Omega$. Thus, by the homotopy property of degree, we get

$$\begin{aligned}
\deg(QN|_{\operatorname{Ker} L}, \Omega \cap \operatorname{Ker} L, 0) &= \deg\bigl(H(\cdot,0), \Omega \cap \operatorname{Ker} L, 0\bigr) \\
&= \deg\bigl(H(\cdot,1), \Omega \cap \operatorname{Ker} L, 0\bigr) \\
&= \deg(\pm J, \Omega \cap \operatorname{Ker} L, 0) \neq 0.
\end{aligned}$$

Then by Theorem 2.1, $Lx = Nx$ has at least one solution in $\operatorname{dom} L \cap \bar{\Omega}$, so that the BVP (1.1), (1.2) has at least one solution in $C^2[0,1]$. The proof is completed. □

By using the same method as in the proof of Lemma 2.2 and Theorem 2.2, we can show the following Lemma 2.3 and Theorem 2.3.

Lemma 2.3. *If $\beta = 1/\eta^2$, $\sum_{i=1}^{m-2} \alpha_i = 1$, $\sum_{i=1}^{m-2} \alpha_i \xi_i^2 = 0$, then $L : \operatorname{dom} L \subset Y \to Z$ is a Fredholm operator of index zero. Furthermore, the linear continuous projector operator $Q : Z \to Z$ can be defined by*

$$Qy = \frac{(l+1)(l+2)(l+3)}{\sum_{i=1}^{m-2} \alpha_i \xi_i^{l+3}} \left(\sum_{i=1}^{m-2} \alpha_i \int_0^{\xi_i} \int_0^{s} \int_0^{\tau} y(v)\,dv\,d\tau\,ds \right) t^l,$$

and the linear operator $K_P : \operatorname{Im} L \to \operatorname{dom} L \cap \operatorname{Ker} P$ can be written by

$$\begin{aligned}
K_P y = {} & \frac{1}{\beta - 1} \left[\int_0^1 \int_0^s \int_0^\tau y(v)\,dv\,d\tau\,ds - \beta \int_0^\eta \int_0^s \int_0^\tau y(v)\,dv\,d\tau\,ds \right] \\
& + \int_0^t \int_0^s \int_0^\tau y(v)\,dv\,d\tau\,ds.
\end{aligned}$$

Also

$$\|K_P\| \leqslant \frac{1}{1-\eta}\|y\|_1, \quad \text{for all } y \in \operatorname{Im} L.$$

Notice that the

$$\operatorname{Ker} L = \bigl\{x \in \operatorname{dom} L:\ x = ct^2,\ c \in R,\ t \in [0,1]\bigr\},$$

$$\operatorname{Im} L = \left\{y \in Z:\ \sum_{i=1}^{m-2} \alpha_i \int_0^{\xi_i}\int_0^{s}\int_0^{\tau} y(v)\,dv\,d\tau\,ds = 0\right\},$$

and

$$\|K_P y\|_\infty \leqslant \frac{1}{\beta-1}\bigl[\|y\|_1 + \beta\eta\|y\|_1\bigr] + \|y\|_1 = \frac{1}{1-\eta}\|y\|_1.$$

Theorem 2.3. *Let $f:[0,1]\times R^3 \to R$ be a continuous function, assume that the condition (A_1) in Theorem 2.2 is satisfied and*:

(A_4) *There exists a constant $M > 0$, such that for $x \in \operatorname{dom} L$, if $|x''(t)| > M$, for all $t \in [0,1]$, then*

$$\sum_{i=1}^{m-2} \alpha_i \int_0^{\xi_i}\int_0^{s}\int_0^{\tau} \bigl[f\bigl(v, x(v), x'(v), x''(v)\bigr) + e(v)\bigr]\,dv\,d\tau\,ds \neq 0.$$

(A_5) *There exists a constant $M^* > 0$, such that for $c \in R$, if $|c| > M^*$, then either*

$$c \cdot \sum_{i=1}^{m-2} \alpha_i \int_0^{\xi_i}\int_0^{s}\int_0^{\tau} \bigl[f\bigl(v, ct^2, 2cv, 2c\bigr) + e(v)\bigr]\,dv\,d\tau\,ds < 0,$$

or else

$$c \cdot \sum_{i=1}^{m-2} \alpha_i \int_0^{\xi_i}\int_0^{s}\int_0^{\tau} \bigl[f\bigl(v, ct^2, 2cv, 2c\bigr) + e(v)\bigr]\,dv\,d\tau\,ds > 0.$$

Then for every $e \in L^1[0,1]$, the BVP (1.1), (1.2) with $\beta = 1/\eta^2$, $\sum_{i=1}^{m-2}\alpha_i = 1$, $\sum_{i=1}^{m-2}\alpha_i\xi_i^2 = 0$ has at least one solution in $C^2[0,1]$ provided that

$$\|a\|_1 + \|b\|_1 + \|c\|_1 < \frac{2-\eta}{1-\eta}.$$

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